

Collective oscillations in two-dimensional Bose-Einstein condensate

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Abstract

We study the effect of lower dimensional geometry on the frequencies of the collective oscillations of a Bose-Einstein condensate confined in a trap. To study the effect of two dimensional geometry we consider a pancake-shaped condensate confined in a harmonic trap and employ various models for the coupling constant depending on the thickness of the condensate relative to the value of the scattering length. These models correspond to different scattering regimes ranging from quasi-three dimensional to strictly two dimensional regimes. Using these models for the coupling parameter and sum rule approach of the many-body response theory we derive analytical expressions for the frequencies of the monopole and the quadrupole modes. We show that the frequencies of monopole mode of the collective oscillations are significantly altered by the reduced dimensionality and also study the evolution of the frequencies as the system make transition from one regime to another.

Recently, several theoretical and experimental studies devoted to the influence of dimensionality on the properties of a Bose-Einstein condensate in a harmonic trap have been reported in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The reduction of dimensionality affects the physical properties of the condensates resulting in very different features from their three dimensional (3D) counterparts. In current experiments on Bose-Einstein condensates with alkali atoms confined in a magnetic trap the anisotropy parameter λ (defined as the ratio between the frequencies of the trap in the z - and the transverse directions) may be varied to achieve condensates with special properties which are characteristic of the low dimensionality. For example, by making the anisotropy parameter much larger than unity ($\lambda \gg 1$) a flatter and flatter (pancake-shaped) condensate can be produced. Such a pancake-shaped condensate is expected to exhibit special features of two dimensional (2D) condensate. It is well known that in the 2D case the scattering properties are very different as compared to the 3D case and this in turn lead to a significant modification of the boson-boson coupling constant. For example, the boson-boson coupling constant in 2D limit becomes density dependent even at a low density and zero temperature. In contrast to this for 3D case, to the lowest order in the density the interactions are described by constant coupling strength and deviations occur only when quantum depletion and finite temperature effects are taken into considerations [17].

In a pancake shaped 3D condensate as the anisotropy parameter is increased the physical properties of the condensate first change due to the modified shape of the confinement [12] and then also due to the alteration of the scattering properties. With respect to the scattering properties of a flat pancake-shaped condensate three regimes can be identified which are characterized by different expressions for the coupling constant [10]. When the linear dimension of the condensate along z -axis given by $a_z = (\hbar/m\omega_z)^{1/2}$ (where ω_z is the z -component of the angular frequency of the trapping potential and m is the mass of the trapped atoms) is still much larger than the 3D scattering length a ($a_z \gg a$), the collisions still take place in three dimensions and this is referred to as quasi-3D (Q3D) regime. On further increasing the anisotropy parameter the condensate gets more tightly confined along the z -axis but the assumption that the scattering is unaffected begins to break down when $a \approx a_z$ and the condensate is said to be in quasi-2D (Q2D) regime. A fully 2D condensate is achieved when $a_z \ll a$ so that the collisions are restricted only in the transverse $\{x, y\}$ plane. It is natural to expect that as the condensate evolves from a fully 3D to a fully

2D regime its static and dynamic properties undergo dramatic changes. In Ref. [10] the evolution of the density profile of a condensate has been investigated as the system crosses from the 3D to the 2D regime. They have found that the widths of the density distribution crucially depend on the collisional properties or the boson-boson interaction parameters in the different regimes.

The main aim of this letter is to study the effect of lower dimensionality on the the frequencies of the collective oscillations of a condensate. In particular we calculate the collective oscillation frequencies of the quadrupole and monopole modes of a flat pancake-shaped condensate and study how these frequencies evolve as the system undergo transitions from the Q3D to the Q2D and to a strictly 2D regime. For this purpose we make use of the sum rule approach of many-body response theory [19, 20]. By employing this method we derive analytical expressions for the frequencies of the quadrupole and the monopole modes. In the following we first derive these expressions and then discuss the results.

We consider a dilute condensate with N bosons confined in an anisotropic (pancake shaped) harmonic trap characterized by the frequencies ω_\perp and $\omega_z = \lambda\omega_\perp$ with the anisotropy parameter λ being much larger than unity. Within the density functional theory, the ground properties of a condensate can be completely described by the ground state condensate density $\rho(\mathbf{r})$ in $\{x, y\}$ plane. The ground state density of the condensate can be determined by minimizing the local density energy functional

$$E[\rho] = \int d^2\mathbf{r} \left[-\frac{\hbar^2}{2m} |\nabla \sqrt{\rho(\mathbf{r})}|^2 + v_{ext}(\mathbf{r})\rho(\mathbf{r}) + \epsilon(\rho)\rho(\mathbf{r}) \right], \quad (1)$$

where $v_{ext}(\mathbf{r})$ is the external harmonic potential in the transverse direction given by

$$v_{ext}(\mathbf{r}) = \frac{1}{2}m\omega_\perp^2 (x^2 + y^2). \quad (2)$$

In the above equation (1) the first and the third terms represent the kinetic energy and the energy due to the interatomic interaction within local density approximation (LDA) respectively. Within this approximation the interaction energy per particle $\epsilon(\rho)$ is given by

$$\epsilon(\rho) = \frac{g}{2}\rho(\mathbf{r}). \quad (3)$$

Where g is the coupling constant whose form depends on the collisional properties of the condensate. For example, g is independent of the density for the 3D case, on the other hand in the purely 2D regime it depends logarithmically on the density. In the following

we briefly describe the models for the coupling constant g valid in the different collisional regimes.

For the 3D system the coupling parameter g which is a constant completely determined by the s-wave scattering length a and it is given by

$$g = \frac{4\pi\hbar^2}{m}a. \quad (4)$$

When the linear dimension a_z of the condensate along the z-direction is much larger the 3D scattering length a ($a_z \gg a$), the collisions still take place in three dimensions. Under this condition the effective coupling constant g_{Q3D} which includes the effects of reduced dimensionality only is given by [4, 12]

$$g_{Q3D} = 2\sqrt{2\pi}\frac{\hbar^2 a}{ma_z} \quad (5)$$

On further increasing the anisotropy and a_z becoming comparable to a ($a_z \approx a$), the collisions start getting affected by the tight confinement along the z-direction. Under such a condition the condensate is said to be in Q2D regime. The coupling constant in this regime is given by

$$g_{Q2D} = \frac{2\sqrt{2\pi}\frac{\hbar^2 a}{ma_z}}{1 + \frac{a}{\sqrt{2\pi}a_z}|\ln(2(2\pi)^{3/2}\rho(\mathbf{r})aa_z)|} \quad (6)$$

Here $\rho(\mathbf{r})$ is the ground state density of the condensate. The above expression was originally derived by Petrov et al. [5, 6] by studying the scattering properties of a bosonic system which is trapped harmonically trapped in the z-direction and uniform in the $\{x, y\}$ plane. A similar expression was later derived by Lee et al. [9] also by employing many-body T-matrix approach. It is important to note here that in the Q2D regime the coupling constant becomes dependent on the density in accordance with the behaviour of collisions in two dimensions. Finally, when a_z becomes much smaller than a ($a_z \ll a$), the collisions can be safely assumed to be taking place in two dimensions resulting in a 2D condensate. The coupling constant in 2D regime is given by

$$g_{2D} = \frac{4\pi\hbar^2}{m} \frac{1}{|\ln\rho(\mathbf{r})a^2|} \quad (7)$$

Notice that the expression for g_{2D} also depends on the density $\rho(\mathbf{r})$ but the information about the confinement direction is absent as it corresponds to the purely 2D case. The expression for g_{2D} was derived in Ref. [18] for a homogenous bose gas of hard disc. This

form of g_{2D} has been employed to study the properties of confined bosons in two dimensions [3, 7] and the rigorous justification for this use was provided by Lieb et al. [8].

Having described the different models for the coupling constant, we now briefly discuss the method for obtaining the frequencies of the monopole and the quadrupole modes of collective oscillations. As mentioned earlier for this purpose we employ the sum-rule approach of the many-body response theory. The most important advantage of this method is that the calculation of frequencies requires the knowledge of the ground-state wave function (or the corresponding ground-state density) of many body system only. In accordance to the basic results of the sum-rule approach [19, 20] the upper bound of the lowest excitation energy is given by

$$\hbar\Omega_{ex} = \sqrt{\frac{m_3}{m_1}} \quad (8)$$

where

$$m_p = \sum_n |\langle 0|F|n\rangle|^2 (\hbar\omega_{0n})^p \quad (9)$$

is the p -th order moment of the excitation energy $\hbar\omega_{0n}$ associated with the excitation operator F and Ω_{ex} is the frequency excitation. Here $\hbar\omega_{n0} = E_n - E_0$ is the excitation energy of eigenstate $|n\rangle$ of the Hamiltonian H of the system. The upper bound given by Eq. (8) is close to the exact lowest excited state when this state is highly collective, that is, when the oscillator strength is almost exhausted by a single mode. This condition is satisfied by the trapped bosons in most of the cases. Moreover, Eq. (8) can be used for computation of the excitation energies by exploiting the fact that the moments m_1 and m_3 can be expressed as expectation values of the commutators between F and H in the ground state $|0\rangle$ [19, 20]:

$$\begin{aligned} m_1 &= \frac{1}{2} \langle 0| [F^\dagger, [H, F]] |0\rangle, \\ m_3 &= \frac{1}{2} \langle 0| [[F^\dagger, H], [[H, [H, F]]] |0\rangle. \end{aligned} \quad (10)$$

For the purpose of calculation of m_1 and m_3 as given by the above equation we need to first choose an appropriate excitation operator F . Following Ref. [13] the excitation operator F is written as

$$F = x^2 + \alpha y^2 \quad (11)$$

with $\alpha = 1$ and $\alpha = -1$ for the monopole and the quadrupole modes respectively. By using the energy functional given by Eq. (1) along with the expression for $\epsilon(\rho)$ we find after

some tedious although straightforward algebra following expressions for the frequencies of the quadrupole

$$\Omega_q = \sqrt{2} \left(1 + \frac{T}{U} \right)^{1/2}, \quad (12)$$

and the monopole

$$\Omega_m = \sqrt{2} \left(1 + \frac{T}{U} + \frac{Y_{int}}{U} \right)^{1/2}, \quad (13)$$

modes. In the above equations T and U denote the kinetic and the harmonic confinement energies and they are given by

$$\begin{aligned} T &= \frac{\hbar^2}{2m} \int d^2\mathbf{r} |\nabla \sqrt{\rho(\mathbf{r})}|^2 \\ U &= \frac{1}{2m} \omega_\perp^2 \int d^2\mathbf{r} (x^2 + y^2) \rho(\mathbf{r}) \end{aligned} \quad (14)$$

On the other hand Y_{int} arises from the interaction energy (third term) term of Eq. (1). As a result of this the values of Y_{int} depend on the model of the coupling constant. The expressions for Y_{int} (in the unit of $N\hbar\omega_\perp$) corresponding to the three regimes are given by

$$\begin{aligned} Y_{int}^{Q3D} &= E_{int}^{Q3D} \\ Y_{int}^{Q2D} &= \left(E_{int}^{Q2D} + 2kI_1^{Q2D} + 2k^2I_2^{Q2D} \right) \\ Y_{int}^{2D} &= \left(E_{int}^{2D} + 2I_1^{2D} - 2I_2^{2D} \right) \end{aligned} \quad (15)$$

with $k = \tilde{a}/\sqrt{\pi}$ and the dimensionless interaction parameter $\tilde{a} = a/a_z$. The general expression for the interaction energy E_{int}^i ($i = Q3D, Q2D, 2D$) is given by

$$E_{int}^i = \int d^2\mathbf{r} \frac{g_i}{2} \rho^2(\mathbf{r}) \quad (16)$$

By using coupling constant given by Eqs. (5)-(7) in the above integral we obtain the interaction energies in the corresponding regime. On the other hand, the expressions for other terms appearing in Eq. (15) can be written as

$$\begin{aligned} I_1^{Q2D} &= \sqrt{2\pi} N \tilde{a} \int \frac{d^2\mathbf{r} \rho^2(\mathbf{r})}{1 + k |\ln (2(2\pi)^{3/2} N \rho(\mathbf{r}) \frac{\tilde{a}}{\lambda})|^2} \\ I_2^{Q2D} &= \sqrt{2\pi} N \tilde{a} \int \frac{d^2\mathbf{r} \rho^2(\mathbf{r})}{1 + k |\ln (2(2\pi)^{3/2} N \rho(\mathbf{r}) \frac{\tilde{a}}{\lambda})|^3} \\ I_1^{2D} &= 2\pi N \int \frac{d^2\mathbf{r} \rho^2(\mathbf{r})}{|\ln (N \rho(\mathbf{r}) \tilde{a}^2)|^2} \\ I_2^{2D} &= 2\pi N \int \frac{d^2\mathbf{r} \rho^2(\mathbf{r})}{|\ln (N \rho(\mathbf{r}) \tilde{a}^2)|^3} \end{aligned} \quad (17)$$

The density appearing in the above equation (Eq. (17)) are normalized to unity and notice that the anisotropy parameter λ is explicitly appearing in the integrals for the Q2D case. Before proceeding further we note that the expressions given by Eqs. (12) and (13) for the Q3D case match with the corresponding results of Ref. [13].

It is evident from Eq. (12) that the frequency of the qudrupole mode is not explicitly dependent on the interaction energy and it is true in all the three regimes considered in this paper . We wish to note here that the Eq. (12) is identical to the expression for the frequency of the quadrupole mode of a 3D condensate with constant coupling parameter [21]. In the absence of inter particle interactions we have $T = U$ and this leads to harmonic oscillator result $\Omega_q = 2$. On the other hand, when interaction energy is much larger than the kinetic energy so that the kinetic energy can be neglected (Thomas-Fermi approximation) we get $\Omega_q = \sqrt{2}$. In the general case one needs to know the value of kinetic energy of the ground state for accurate estimation of the frequency of quadrupole mode.

Now we focus our attention on the frequency of the monopole mode of the collective oscillations. In contrast to the quadrupole case the frequencies of monopole mode are explicitly dependent on the interaction energy. Therefore, the frequencies of the monopole mode will be affected by different models of the coupling constant. To illustrate the dependence of the interaction energy on the frequencies more clearly we substitute the virial relations associated with the three models

$$\begin{aligned} T - U + E_{int}^{Q3D} &= 0 \\ T - U + E_{int}^{Q2D} + kI_1^{Q2D} &= 0 \\ T - U + E_{int}^{2D} + I_1^{2D} &= 0 \end{aligned} \tag{18}$$

in the respective expressions in Eq. (13) to obtain

$$\begin{aligned} \Omega_m^{Q3D} &= 2 \\ \Omega_m^{Q2D} &= 2 \left(1 + k \frac{I_1^{Q2D}}{2U} + k^2 \frac{I_2^{Q2D}}{U} \right)^{1/2} \\ \Omega_m^{2D} &= 2 \left(1 + \frac{I_1^{Q2D}}{2U} - \frac{I_2^{Q2D}}{U} \right)^{1/2} \end{aligned} \tag{19}$$

The first of the above equation shows that unlike the Q2D and the 2D cases the frequency of the monopole mode in the Q3D regime is independent of the coupling constant. The

frequency of the monopole mode in the Q3D regime is given by the frequency of a 2D harmonic oscillator. It is important to point out that in contrast to the Q3D case the frequency of the monopole mode of a 3D condensate is explicitly dependent on the interaction strength [21].

Now we turn to the detail study of the frequencies of the monopole mode for the Q2D and the 2D models of the coupling constant. To this end we first need to evaluate the integrals given in the Eq. (17). For this purpose we employ the ground state densities $\rho(\mathbf{r})$ of the condensates which are obtained within the Thomas-Fermi (TF) approximation. Furthermore, to obtain the densities within the TF approximation the spatial dependence of the coupling constants is also neglected by using results of the homogenous system to relate the density to the chemical potential. It has been shown in the Refs. [9, 10] that for large N the TF approximation and spatially independent form of the coupling constant yield sufficiently accurate results.

We begin the discussion of the results with the values of relevant parameters from the experiments of Gorlitz et al. [15] with ^{23}Na atoms. These parameters are $N = 10^5$, $\lambda = 26.33$ and $\tilde{a} = 3.8 \times 10^{-3}$. We note here that this value of N is consistent with the TF approximation. These parameters indicate that the condensate produced in the experiment falls within the Q3D regime. The numbers obtained by us for the monopole frequencies of this system are $\Omega_{Q2D} = 2.001$ and $\Omega_{2D} = 2.084$. These results clearly show that for the above system the Q2D result is very close to the corresponding Q3D number. However, the 2D model overestimates the frequency of the monopole mode and this is anticipated as this model is not applicable to the condensate achieved in the above-mentioned experiment.

Now to study the effect of different models of the coupling constant and their applicability we choose three different values of the parameter \tilde{a} : $\tilde{a} = 3.8 \times 10^{-3}$, 0.33 and 2.68. These values are chosen such a way that the first, the second and the third numbers fall in the Q3D, the Q2D and the 2D regimes respectively. In addition to this we also choose $\lambda = 2 \times 10^5$ so that the condensate has negligible length in the z -direction and the motion along this axis is completely frozen. The results with these parameters are presented in Table I. Again we can see from Table I that for $\tilde{a} = 3.8 \times 10^{-3}$ the numbers predicted by the Q3D and the Q2D models are very close and the corresponding result from the 2D model is quite higher than these two numbers. For $\tilde{a} = 0.33$ the numbers obtained by the both Q2D and 2D models are markedly different from the result of Q3D model. As has been discussed before

for $\tilde{a} = 0.33$ along with the large value of λ the scattering properties start to get influenced by the confinement. In this situation it is expected that the Q2D and 2D models will give significantly different numbers in comparison to the corresponding Q3D result. In the light of our earlier discussion we expect that for $\tilde{a} = 0.33$ the coupling constant is better described by the Q2D model. On the other hand, for this value of \tilde{a} the frequency of the monopole mode obtained with the 2D model is higher than that of Q2D model. As the interaction parameter \tilde{a} is further increased to a value $\tilde{a} = 2.68$ the scattering properties become truly two dimensional, consequently for this value of \tilde{a} the 2D model should be able to predict the frequency of the monopole mode of a two dimensional condensate. In contrast to the case of $\tilde{a} = 0.33$, for $\tilde{a} = 2.68$ the value of the frequency obtained by the 2D model is lower than the corresponding number from the Q2D model.

Finally, for the sake of completeness we plot in Fig. 1 the frequencies of the monopole mode obtained with three different models as a function of the interaction parameter \tilde{a} . The curves are drawn with the anisotropy parameter $\lambda = 5 \times 10^5$ and the number of atoms $N = 10^5$. It can be clearly seen from Fig. 1 that the frequencies of the monopole mode obtained with the three models of the coupling constant exhibit different trend with the increase in the interaction parameter \tilde{a} . For example, in contrast to the constant value of the frequency for the Q3D case the frequencies obtained by employing the Q2D and the 2D models increase as \tilde{a} is increased. As mentioned before it is only for $\tilde{a} \ll 1$ the values of frequencies obtained by the Q3D and the Q2D models are very close and the 2D model gives quite different numbers. On the other hand, when \tilde{a} exceeds the value 10^{-2} , the effects of reduced dimensionality start affecting the scattering properties and both Q2D and 2D models give different results as compared to the Q3D numbers. Therefore, we conclude from our results that the collective frequencies of the monopole mode and their behaviour can be used to identify the dimensionality of the systems as the system evolves from Q3D to a strictly 2D regime. The results of theoretical calculations can also be used to test the validity of different models of the coupling constant by comparing them with the experimental values.

In summary, we have calculated the frequencies of collective oscillations of the Bose-Einstein condensate confined in a flat pancake-shaped trap. The condensate is tightly trapped along z-axis such that the motion along this axis is frozen. For such a condensate depending on the value of the ratio between the size of the condensate along the z-direction and the s-wave scattering length, three different regimes can be identified. These

three different regimes are described by three different models for the boson-boson interactions. We have calculated the frequencies of the collective oscillations corresponding to these three models. For this purpose we have used sum-rule approach of many body response theory along with the ground-state density obtained within the TF approximation. The main result of this paper is that the different models for the coupling constant are clearly manifested in the frequency of the monopole mode and they lead to distinct results in the region of their applicability. The effect of modification of the collision properties in two dimension start changing the monopole frequency when the value of \tilde{a} becomes more than 10^{-2} . On the other hand, quadrupole mode is not explicitly dependent on the coupling parameter and thus not affected by the boson-boson interactions.

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Figure captions

Fig.1Frequencies (in units of ω_{\perp}) of the monopole mode of 5×10^5 ^{23}Na atoms confined in a highly deformed trap with $\lambda = 2 \times 10^5$ as a function of the interaction parameter \tilde{a} . The solid line represents results for Q2D case while the corresponding 2D results are shown by the dashed line and the Q3D results are displayed by the horizontal line.

TABLE I: Frequencies of the monopole mode in the units of ω_{\perp} for three different values of the dimensionless interaction parameter \tilde{a} calculated using Eq. 19 for $N = 5 \times 10^5$ and $\lambda = 2 \times 10^5$

| \tilde{a} | Ω_{Q3D} | Ω_{Q2D} | Ω_{2D} |
|----------------------|----------------|----------------|---------------|
| 3.8×10^{-3} | 2.00 | 2.001 | 2.037 |
| 0.33 | 2.00 | 2.049 | 2.084 |
| 2.68 | 2.00 | 2.292 | 2.182 |

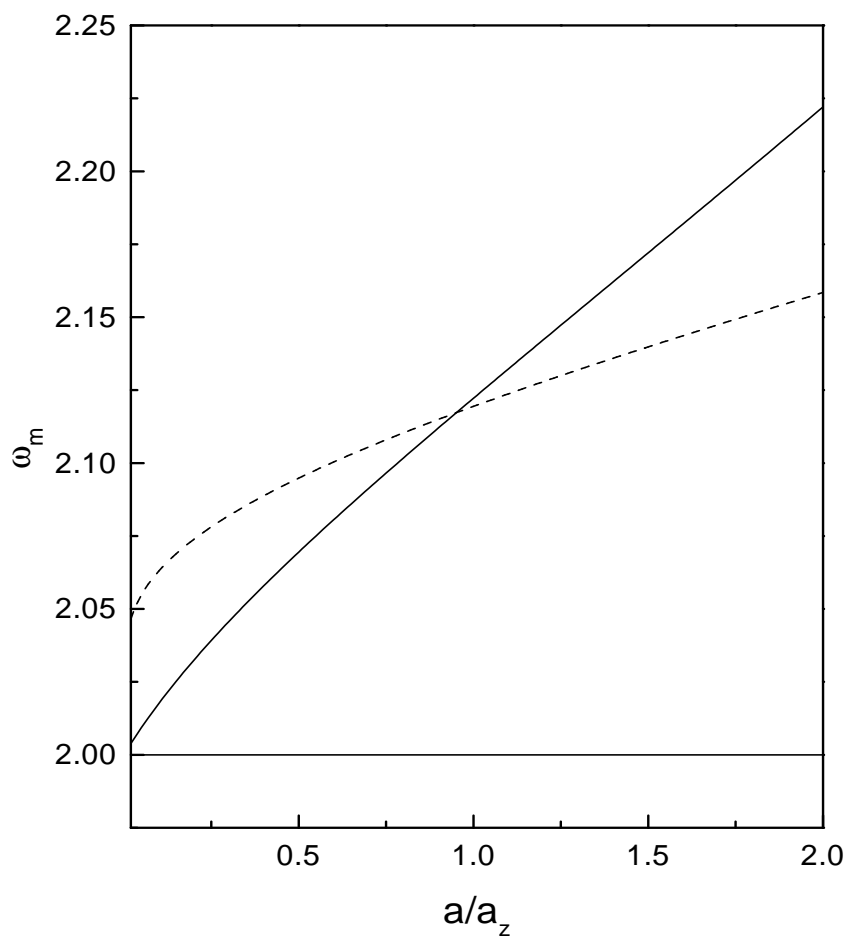


FIG. 1: Caption rotates along the figure.